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The passage through resonance of synchronous electric motors mounted on an elastic base☆

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ABSTRACT

It is proved that the Sommerfeld effect does not occur for any passage through resonance for a synchronous electric motor with an asynchronous start-up, mounted on an elastic base.

1. Introduction

This paper was stimulated by the paper[1] where "jamming" of an electric motor on an elastic base was observed in the neighbourhood of a resonance of the oscillation frequency. This jamming occurs in the form of chaotic oscillations[1] and is therefore often called the Sommerfeld effect.[1–7]

The equation[1–7]

$$I\ddot{\varphi} = G(\dot{\varphi}) \tag{1.1}$$

is the generally accepted mathematical model of an electric motor in the study of the Sommerfeld effect. Here, $\varphi(t)$ is the phase and $\dot{\varphi}(t)$ is the instantaneous frequency of rotation of the rotor of the electric motor, $G(x)$ is the so-called "characteristic of the electric motor minus the load" and I is the moment of inertia of the rotor.

With natural assumptions, it is shown below that the Sommerfeld effect does not occur for any passage through resonance of a synchronous electric motor with an asynchronous start-up. This fact is based on a combined treatment of the motions of the elastic base and mathematical models of synchronous electric motors which are more complex than (1.1). At the same time, it is shown that synchronous motors possess internal stabilizing properties which prevent the Sommerfeld effect.

Synchronous electrical machines are used in naval vessels as electric motors and power plants. Jamming in the neighbourhood of a resonance during start-up leads to oscillations of the elastic base with appreciable amplitudes and thereby to the detection of the vessel. The asymptotic estimates of the amplitude of the elastic base oscillations which are obtained below enable one to determine the parameters of the system in order to avoid such detection.

2. Boundedness of the solutions of the equations for a synchronous motor on an elastic base

We recall the classical equations for an electric motor - elastic base system in the case of the model (1.1)[1,3,6] (Fig. 1)

$$I\ddot{\varphi} = G(\dot{\varphi}) + m\varepsilon\ddot{z}\sin\varphi, \quad M\ddot{z} + \beta\dot{z} + cz = -m\varepsilon(\cos\varphi)^{\bullet\bullet} \tag{2.1}$$

Here φ is the phase of the rotation of the rotor, z is the deviation of the platform from the equilibrium state, M is the mass of the platform, m is the mass of the rotor, I is the moment of inertia of the rotor, c and β are coefficients of elasticity and viscous friction and ε is the eccentricity.

Adopting the equation[8–11]

$$I\ddot{\theta} = -\alpha\dot{\theta} - \sin\theta$$

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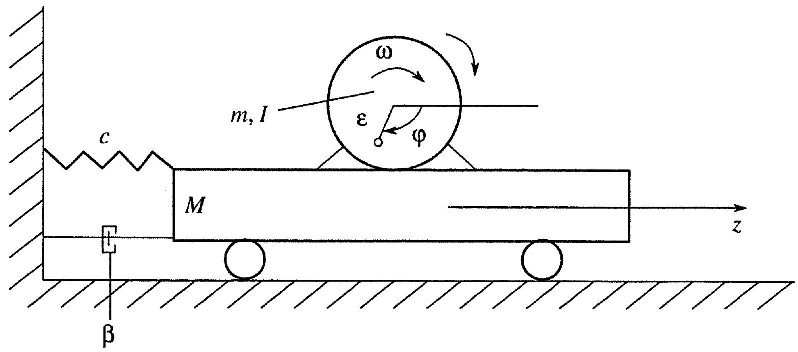

Fig. 1.

as the mathematical model of a synchronous electric motor, we similarly have the equations

$$I\ddot{\theta} = -\alpha\dot{\theta} - \sin\theta + m\varepsilon\ddot{z}\sin(\omega t + \theta)$$
$$M\ddot{z} + \beta\dot{z} + cz = -m\varepsilon(\cos(\omega t + \theta))^{\bullet\bullet} \tag{2.2}$$

for the synchronous motor - elastic base system. Here ω is the frequency of the current in the windings of the stator, θ is the difference between the phases of the rotating magnetic field and the rotor[10,11] and α is the damper winding coefficient. For asynchronous start-up of the motor, we have the equalities

$$z(0) = 0, \quad \dot{z}(0) = 0, \quad \dot{\theta}(0) = 0 \tag{2.3}$$

We now make the replacement

$$z = u - \kappa\cos(\theta + \omega t), \quad \kappa = m\varepsilon/M$$

From the second equation of system (2.2), we obtain the relation

$$M\ddot{u} + \beta\dot{u} + cu = \beta\kappa(\cos(\theta + \omega t))^{\bullet} + c\kappa(\cos(\theta + \omega t)) \tag{2.4}$$

and the estimates

$$|u(0)| \le \kappa, \quad |\dot{u}(0)| \le |\kappa\omega| = \kappa\omega \tag{2.5}$$

follow from the first and the last two of equalities (2.3) respectively.

From Eq. (2.4) and estimates (2.5), we obtain the inequality $|u(t)| \le Q\varepsilon$, $\forall t \ge 0$ from which the estimate

$$|z(t)| \le D\varepsilon, \quad \forall t \ge 0 \tag{2.6}$$

follows. Here, Q and D are certain numbers which depend on the parameters β, c, m, M and ω.

System (2.2) is equivalent to the system

$$I\Theta(\omega t + \theta)\ddot{\theta} = -\alpha\dot{\theta} - \sin\theta + \frac{m\varepsilon}{M}(\sin(\omega t + \theta))(-\beta\dot{z} - cz + m\varepsilon(\cos(\omega t + \theta))(\omega + \dot{\theta})^2)$$
$$M\Theta(\omega t + \theta)\ddot{z} = -\beta\dot{z} - cz + m\varepsilon((\dot{\theta} + \omega)^2\cos(\omega t + \theta)) + \frac{m\varepsilon}{I}\sin(\omega t + \theta)(-\alpha\dot{\theta} - \sin\theta) \tag{2.7}$$

Here

$$\Theta(\omega t + \theta) = 1 - \frac{(m\varepsilon)^2}{IM}\sin^2(\omega t + \theta)$$

We next consider the function $V = \dot{\theta}^2 + \dot{z}^2$ and those values of $\dot{\theta}$ and $\dot{z}$ for which

$$V = R \tag{2.8}$$

It is clear that the inequalities

$$|\dot{\theta}| \le \sqrt{R}, \quad |\dot{z}| \le \sqrt{R} \tag{2.9}$$

follow from this.

For any arbitrary function V along the trajectories of system (2.7), we have the relation

$$\frac{1}{2}\dot{V} = \frac{\dot{\theta}}{I\Theta(\omega t + \theta)}\left[-\alpha\dot{\theta} - \sin\theta + \frac{m\varepsilon}{M}\sin(\omega t + \theta)(-\beta\dot{z} - cz + m\varepsilon(\cos(\omega t + \theta))(\omega + \dot{\theta})^2)\right] +$$
$$+ \frac{\dot{z}}{M\Theta(\omega t + \theta)}\left[-\beta\dot{z} - cz + m\varepsilon((\dot{\theta} + \omega)^2\cos(\omega t + \theta)) + \frac{m\varepsilon}{I}\sin(\omega t + \theta)(-\alpha\dot{\theta} - \sin\theta)\right]$$

It follows from estimates (2.6) and (2.9) and relation (2.8) that the estimate

$$\dot{V} \le -\delta R + E\varepsilon R^2$$

is satisfied for sufficiently large R and small ε. Here, δ and E are certain positive numbers which depend on the parameters β, c m, M, ω, I and α.

It is clear that the inequality

$$\dot{V} < 0$$

follows from this estimate for sufficiently small $\varepsilon \in [0, \varepsilon_0]$, $\varepsilon_0 = \varepsilon_0(R)$.

It follows from this and from equality (2.8) that, in the case of the solution of system (2.2) with initial data (2.3) considered, the inequality

$$\dot{z}(t)^2 + \dot{\theta}(t)^2 \le R, \quad \forall t \ge 0$$

is satisfied for sufficiently large R (relative to the parameters I, M, m, β, c, α and ω) and small $\varepsilon \in [0, \varepsilon_0(R)]$.

The following result has therefore been proved.

Theorem 1. *For sufficiently small $\varepsilon > 0$, the solution $\theta(t)$, $z(t)$ with initial data (2.3) and sufficiently large L, satisfies the estimate*

$$|\dot{\theta}(t)| \le L, \quad |z(t)| \le L, \quad |\dot{z}(t)| \le L, \quad |\ddot{z}(t)| \le L, \quad \forall t \ge 0$$

3. Asymptotic estimates of the solutions of a pendulum-type equation for small unsteady perturbations

We now consider the equation

$$\ddot{x} + a\dot{x} + \sin x = p(t); \quad |p(t)| \le \varepsilon, \quad \forall t \in R^1 \tag{3.1}$$

where a is a positive number, $p(t)$ is a continuous function which satisfies the above-mentioned condition, ε is a certain positive number which we subsequently assume to be small with respect to a and 1: $\varepsilon \ll a$, $\varepsilon \ll 1$.

Theorem 2. *For any solution of Eq. (3.1), an integer k exists such that*

$$\overline{\lim_{t \to +\infty}} |x(t) + k\pi| \le C\varepsilon \tag{3.2}$$

where C satisfies the relations

$$C > 1 \text{ when } a \ge 2$$

$$C > \frac{1 + P}{1 - P}, \quad P = \exp\left(-\frac{a\pi}{\sqrt{4 - a^2}}\right) \text{ when } a < 2$$

Proof. Consider the following system, which is equivalent to Eq. (3.1),

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -ax_2 - \sin x_1 + p(t) \tag{3.3}$$

and the so-called comparison systems[12]

$$\dot{x}_2 = x_2, \quad \dot{x}_2 = -ax_2 - \sin x_1 - \varepsilon \tag{3.4}$$

$$\dot{x}_2 = x_2, \quad \dot{x}_2 = -ax_2 - \sin x_1 + \varepsilon \tag{3.5}$$

It is well known[12] that, for sufficiently small ε, all of the solutions of system (3.4) and (3.5) are bounded in the time interval $(0, +\infty)$ and, consequently,[12] solutions $F_k^1(\sigma)$ and $F_k^2(\sigma)$ of the equations

$$F^j dF^j/d\sigma + aF^j + \sin\sigma = (-1)^j\varepsilon, \quad j = 1, 2 \tag{3.6}$$

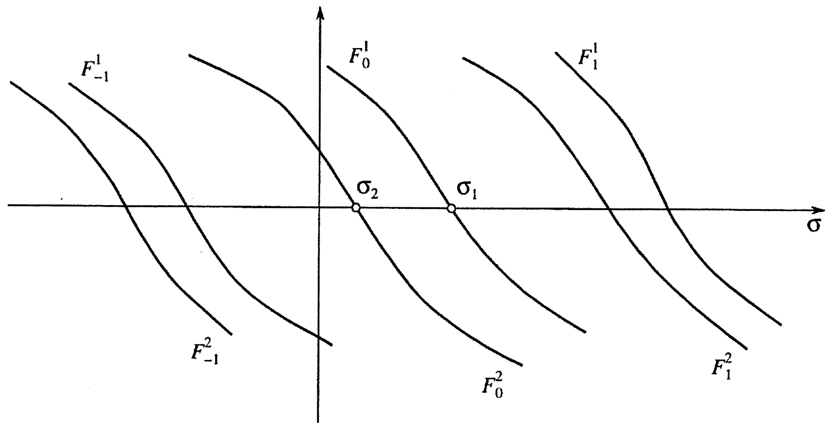

Fig. 2.

exist which satisfy the relations (Fig. 2)

$$F_k^j(\sigma_j + 2\pi k) = 0$$
$$F_k^j(\sigma) < 0, \quad \forall\sigma > \sigma_j + 2k\pi; \quad F_k^j(\sigma) > 0, \quad \forall\sigma < \sigma_j + 2k\pi$$
$$\lim_{\sigma\to\infty} \left|F_k^j(\sigma)\right| = +\infty \tag{3.7}$$

Here σ_1 and σ_2 are zeroes of the functions $\sin\sigma + \varepsilon$ and $\sin\sigma - \varepsilon$ respectively in the set $[0, 2\pi)$ such that $\cos\sigma_1 < 0$ and $\cos\sigma_2 < 0$.

We now consider the solution $x_1(t), x_2(t)$ of system (3.3) which, for a certain t, satisfies the equation

$$x_2(t) = F_k^2(x_1(t)) > 0$$

It is obvious that

$$\frac{\dot{x}_2(t)}{\dot{x}_1(t)} = -\frac{aF_k^2(x_1(t)) - \sin x_1(t) + p(t)}{F_k^2(x_1(t))} < \frac{-aF_k^2(x_1(t)) - \sin x_1(t) + \varepsilon}{F_k^2(x_1(t))} = \left.\frac{dF_k^2(x)}{dx}\right|_{x = x_1(t)}$$

Consequently, the curve $x_2 = F_k^2(x_1), x_2 > 0$ is transverse to the vector field of system (3.3) and the solution $x_1(t), x_2(t)$ intersects this curve as it passes downwards (Fig. 3).

It is similarly proved that the curve $x_2 = F_k^1(x_1), x_2 < 0$ is transverse to the vector field of system (3.3) and, as it passes downwards, the solution $x_1(t), x_2(t)$ intersects this curve (Fig. 3).

Here there is therefore the family of closed transverse curves which is shown in Fig. 3. It depends on the integral parameter k, and, for any point of the space $\{x_1, x_2\}$, a closed transverse curve is found containing this point within it. The boundedness of any solution of system (3.3) in the time interval $(0, +\infty)$ follows directly from this. Moreover, it is clear from Fig. 3 and the relation $\dot{x}_1(t) = x_2(t)$ that, for any solution $x_1(t), x_2(t)$, either a pair of numbers $\tau > 0$ and k exists such that

$$x_2(\tau) = 0, \quad x_1(\tau) \in (\sigma_2 + 2k\pi, \sigma_1 + 2(k+1)\pi)$$

or a k exists such that

$$\lim_{t\to+\infty} x_2(t) = 0, \quad \lim_{t\to+\infty} x_1(t) \in [\sigma_2 + 2k\pi, \sigma_1 + 2(k+1)\pi]$$

Hence it follows that, for any solution $x_1(t), x_2(t)$, a number $T > 0$ exists which satisfies the following condition.

When $t \geq T$, in the case of a certain k the solution $x_1(t), x_2(t)$ belongs to the set Φ_k (Fig. 4):

$$\Phi_k = \Omega_k \cup \Psi_k \cup \Omega_{k+1}$$

$$\Omega_k = \{x_1 \in [\sigma_1 + 2k\pi, \sigma_2 + 2(k+1)\pi], F_k^1(x_1) \leq x_2 \leq F_{k+1}^2(x_1)\}$$

$$\Psi_k = \{x_1 \in [\sigma_2 + 2(k+1)\pi, \sigma_1 + 2(k+1)\pi], \tilde{F}_{k+1}^1(x_1) \leq x_2 \leq \tilde{F}_{k+1}^2(x_1)\}$$

Here, $\tilde{F}_k^1(\sigma)$ and $\tilde{F}_k^2(\sigma)$ are solutions of Eq. (3.6) which satisfy the conditions

$$\tilde{F}_k^j(\sigma_j + 2k\pi) = 0, \quad j = 1, 2$$

$$\tilde{F}_k^1(\sigma) < 0, \quad \tilde{F}_k^2(\sigma) > 0, \quad \forall\sigma \in (\sigma_2 + 2k\pi, \sigma_1 + 2k\pi)$$

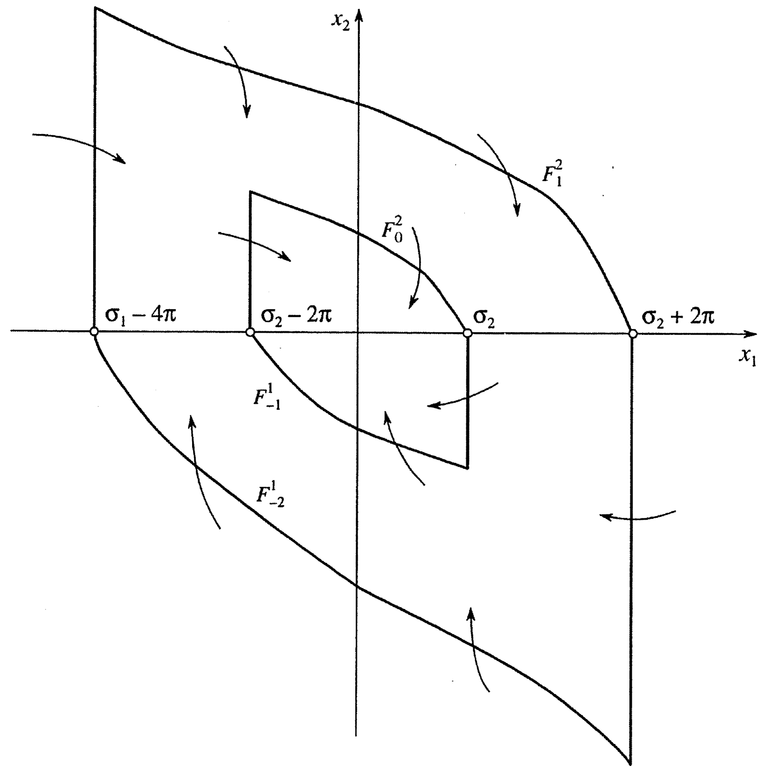

Fig. 3.

It is clear from Fig. 4 that the sets Φ_k are positively invariant. Furthermore, the solution $x_1(t), x_2(t)$ is either found for all $t \geq T$ in the set Ψ_k or a $\tau \geq T$ exists such that this solution is found in Ω_k or Ω_{k+1} for all $t \geq \tau$.

In the first case,

$$\overline{\lim_{t \to +\infty}} \left| x_1(t) + \frac{\sigma_1 + \sigma_2}{2} + 2k\pi \right| \leq |\sigma_1 - \sigma_2| \tag{3.8}$$

Since, for small ε,

$$\sigma_1 = \pi + \varepsilon + o(\varepsilon), \quad \sigma_2 = \pi - \varepsilon + o(\varepsilon)$$

relation (3.2) for $C > 1$ follows from inequality (3.8).

We will now consider the case when the solution $x_1(t), x_2(t)$ is located in the domain Ω_k for all $t \geq \tau$ and construct the continuous family of transverse curves.

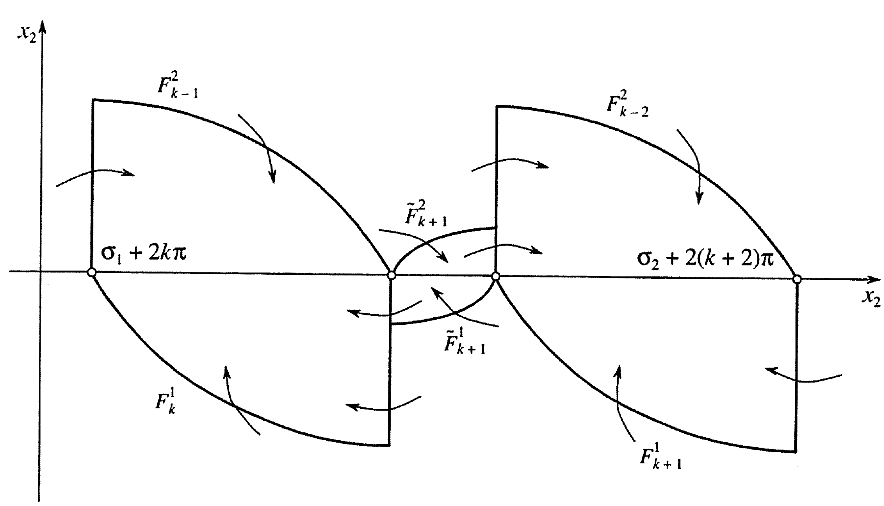

Fig. 4.

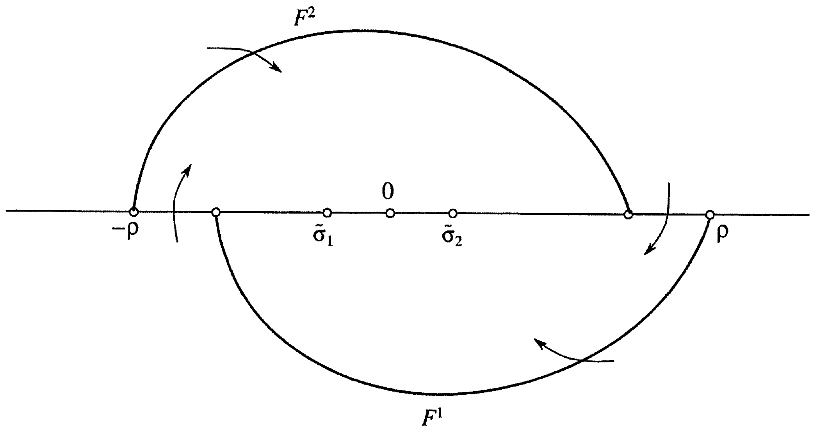

Fig. 5.

Suppose $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are zeroes of the functions $\sin\sigma+\varepsilon$ and $\sin\sigma-\varepsilon$ in the set $[-\pi, \pi)$ respectively such that $\cos\tilde{\sigma}_1 > 0$ and $\tilde{\sigma}_2 > 0$. It is clear that $\tilde{\sigma}_1 = -\varepsilon + o(\varepsilon)$, $\tilde{\sigma}_2 = \varepsilon + o(\varepsilon)$. Without loss in generality, we shall consider the case Ω_{-1}.

We now introduce the parameter $\rho \ge \rho_0 \ge \tilde{\sigma}_2$ into the treatment and choose the number ρ_0 such that the solution of Eq. (3.6) when $j=1$ with initial data $F^1(\rho)=0$ in the half-plane $\{F^1 \le 0\}$ possesses the following property

$$F^1(\sigma) < 0, \quad \forall\sigma \in (-\nu, \rho), \quad F^1(-\nu) = 0, \quad -\nu > -\rho$$

The solution of Eq. (3.6) when $j=2$ with the initial data $F^2(-\rho)=0$ in the half-plane $\{F^2 \ge 0\}$ will then similarly possess the following property

$$F^2(\sigma) > 0, \quad \forall\sigma \in (-\rho, \nu), \quad F^2(\nu) = 0, \quad \nu < \rho$$

Here, it is easily proved, as was done earlier, that the curves $x_2=F^1(x_1)$, $x_2=F^2(x_1)$ are transverse for system (3.3) (Fig. 5).

Hence, a family of transverse closed curves $\gamma(\rho)$ of the form

$$x_2 = \begin{cases} F^1(x_1), & x_1 \in (-\nu, \rho) \\ 0, & x_1 \in (-\rho, -\nu) \end{cases}, \quad x_2 = \begin{cases} F^2(x_1) & x_1 \in (-\rho, \nu) \\ 0, & x_1 \in (\nu, \rho) \end{cases}$$

has been constructed in the set Ω_{-1}.

It follows from this that, in the case of a solution $x_1(t)$, $x_2(t)$ of system (3.3) located in the set Ω_{-1}, a number τ exists such that, when $t \ge \tau$, this solution will lie in the domain bounded by the curve $\gamma(\rho)$, $\rho > \rho_0$.

We now determine the number ρ_0, using the smallness of ε.

Using the linearization of the comparison systems (3.4) and (3.5) of the equivalent equations (3.6), when $j=1$ and $j=2$ we immediately obtain that, when $a \ge 2$,

$$\rho_0 = \sigma_2 \tag{3.9}$$

When $a<2$, for the linearized system we have

$$\rho_0 = \frac{1+P}{1-P}\varepsilon \tag{3.10}$$

Hence, for small $\varepsilon > 0$,

$$\overline{\lim_{t\to+\infty}} |x_1(t)| \le \rho < \rho_0$$

From this and from equalities (3.9) and (3.10), we obtain estimate (3.2).

Note that Theorem 2 can be extended to different two-dimensional non-autonomous non-linear systems with a cylindrical phase space in the spirit of the papers.[11–16]

It follows from estimate (3.2) that the inequality

$$\overline{\lim_{t\to+\infty}} |\sin x(t)| \le C\varepsilon$$

holds for small ε. The following result can be obtained from this and from the condition for the function $p(t)$ (3.1).

Theorem 3. *The estimates*

$$\overline{\lim_{t\to+\infty}} |\dot{x}(t)| \le \frac{C+1}{a}\varepsilon, \quad \overline{\lim_{t\to+\infty}} |\ddot{x}(t)| \le 2(C+1)\varepsilon \tag{3.11}$$

are satisfied for any solution of Eq. (3.1).

4. Proof that there is no Sommerfeld effect

We now apply Theorems 1–3 to system (2.2) by putting

$$t = \tau\sqrt{I}, \quad p(\tau) = m\varepsilon I^{-1}\ddot{z}\sin(\sqrt{I}\omega\tau + \theta)$$

It follows from Theorem 1 that, for any solution of system (2.2) with initial data (2.3), the following relation is satisfied for a certain k

$$|p(t)| \le m\varepsilon I^{-1}L, \quad \forall t \ge 0$$

However, according to Theorems 2 and 3, we then obtain the estimates

$$\overline{\lim_{t\to+\infty}}\,|\theta(t) + k\pi| \le Cm\varepsilon I^{-1}L, \quad a = \alpha I^{-1/2}$$

$$\overline{\lim_{t\to+\infty}}\,|\dot\theta(t)| \le (C+1)m\varepsilon\alpha^{-1}I^{-1/2}L, \quad \overline{\lim_{t\to+\infty}}\,|\ddot\theta(t)| \le 2(C+1)m\varepsilon I^{-1}L$$

Using these estimates and applying elementary transformations to the second equation of system (2.2), we obtain the relation

$$M\ddot{z} + \beta\dot{z} + cz = m\varepsilon\omega^2\cos\omega t + O(\varepsilon^2)$$

which is satisfied for large t.

It follows from this that, after a transient process, operating conditions are established in a synchronous machine with a rotation frequency of the rotor $\omega + O(\varepsilon)$, and the oscillations of the elastic base are harmonic with frequency $\omega + O(\varepsilon)$ and amplitude

$$\frac{m\varepsilon\omega^2}{|M\omega^2 - c - \beta i\omega|} + O(\varepsilon^2)$$

Hence, if the natural frequency of the elastic base is less than ω, then, in the case of an asynchronous start-up, the synchronous machine - elastic base system always slips through a resonance in the transient process and reaches synchronous conditions.

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